14

CHAPTER

Taylor's and Laurent's Series

14.1 INTRODUCTION

An analytic function within a circle can be expanded by Taylor's series.

If a function which is not analytic within a circle is expanded by Laurent's series.

14.2 CONVERGENCE OF A SERIES OF COMPLEX TERMS

Let $(u_1 + iv_1) + (u_2 + iv_2) + (u_3 + iv_3) + ... + (u_n + iv_n) + ...$ ... (1)

be an infinite series of complex terms: $u_1, u_2, ... v_1, v_2 ...$ being real numbers.

(a) If the series Σu_n and Σu_n converge to the sums U and V then series (1) is said to converge to the sum $U + iV$.

(b) If (1) is a convergent series, then

$$\lim_{n\to\infty} (u_n + iv_n) = 0$$

(c) The series (1) is said to be **absolutely convergent** if the series

$$|u_1 + iv_1| + |u_2 + iv_2| + |u_3 + iv_3| + ... + |u_n + iv_n| + ...$$

is convergent. Since $|u_n|$ and $|v_n|$ are both less than $|u_n + iv_n|$.

(d) Let the series

$$a_1(z) + a_2(z) + a_3(z) + ... + a_n(z) + ... \quad ...(2)$$

converge to the sum $S(z)$ and $S_n(z)$ be the sum of its first n terms.

The series (2) is said to be absolutely convergent in region R, if corresponding to any positive number ε, there exists a positive number N.

$$|S(z) - S_n(z)| < \varepsilon \text{ for } n > N$$

(e) Weirstras's, M-test holds good for series of complex terms also.

Series (2) is uniformly convergent in a region R if there is a convergent series ΣM_n. Such that $|a_n(z)| \le M_n$

A uniformly convergent series can be integrated term by term.

14.3 POWER SERIES

A series in powers of $(z - z_0)$ is called power series.

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + ... + a_n(z - z_0)^n + ... \quad ...(1)$$

Here $a_0, a_1, a_2, ..., a_n, ...$ are known as the coefficients of the series.

Here z is a complex variable and z_0 is called the centre of the series.

(1) is also called the power series about the point z_0.

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n + ...$$

Here the centre of the series is zero.

14.4 REGION OF CONVERGENCE

The region of convergence is the set of all points z for which the series converges.

There are three distinct possibilities for a convergent series.

(1) The series converges only at the point $z = z_0$

(2) The series converges for all the points in the whole plane.

(3) The series converges everywhere inside a circular plane $|z - z_0| < R$, where R is the radius of convergence and diverges everywhere outside the circle/circular ring.

14.5 RADIUS OF CONVERGENCE OF POWER SERIES

Consider the power series $\Sigma a_n z^n$.

By Cauchy theorem on limits, radius of convergence R is given by

(i) $\dfrac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$ (ii) $\dfrac{1}{R} = \lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$

Example 1. *Find the radius of convergence of the power series*

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Solution. Here, $a_n = \dfrac{1}{n!} \Rightarrow a_{n+1} = \dfrac{1}{(n+1)!}$

Radius of convergence is given by

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

$$R = \infty$$

Hence, the radius of convergence of the given power series is ∞. **Ans.**

Example 2. *Find the radius of convergence of the power series*

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^n + 3}$$

Solution. Here,

$$a_n = \frac{1}{2^n + 3}$$

$$\Rightarrow \quad a_{n+1} = \frac{1}{2^{n+1} + 3}$$

Radius of convergence is given by

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^n+3}{2^{n+1}+3} = \lim_{n\to\infty}\frac{1+\frac{3}{2^n}}{2+\frac{3}{2^n}} = \frac{1}{2}$$

$$R = 2$$

Hence, the radius of convergence of the given power series is 2. **Ans.**

Example 3. *Find the radius of convergence of the power series:*

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n^n}$$

Solution. Here,

$$a_n = \frac{1}{n^n}, \quad a_{n+1} = \frac{1}{(n+1)^{n+1}}$$

Radius of convergence is given by

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty}\frac{n^n}{(n+1)^n (n+1)} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n (n+1)} = 0$$

$$\Rightarrow \quad R = \infty$$

Hence, the radius of convergence of the given power series is ∞. **Ans.**

Example 4. *Find the radius of convergence of the power series:*

$$f(z) = \sum_{n=0}^{\infty} \frac{n!}{n^n} z^n$$

Solution. Here, $a_n = \dfrac{n!}{n^n}$

$$\therefore \quad a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

Now,

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!} = \frac{n!}{(n+1)^n}\cdot\frac{n^n}{n!} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

Radius of convergence is given by

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$$

$$\Rightarrow \quad R = e.$$

Hence, the radius of convergence of the given power series is e. **Ans.**

EXERCISE 14.1

Find the radius of convergence of following power series:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^p}$ **Ans.** 1

2. $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}}$ **Ans.** 3

3. $\displaystyle\sum_{n=0}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} z^n$ **Ans.** $\dfrac{1}{e}$

4. $\displaystyle\sum_{m=0}^{\infty} (5+12i)^m z^m$ **Ans.** 13

5. $\displaystyle\sum_{n=0}^{\infty} \frac{2n+3}{(2n+5)(n+5)} z^n$ **Ans.** 1

14.6 METHOD OF EXPANSION OF A FUNCTION

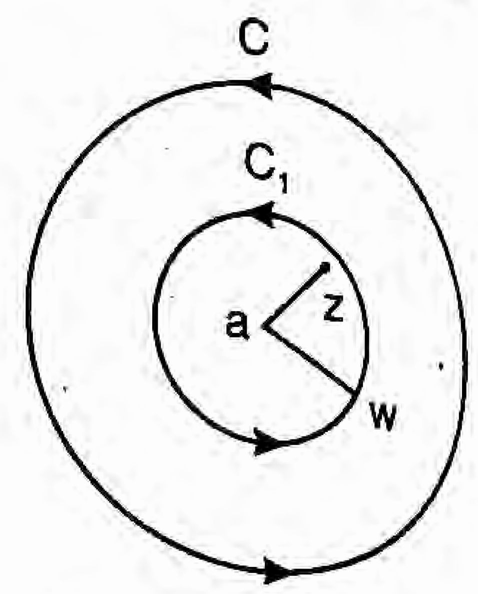
(1) Taylor's series
(2) Binomial series
(3) Exponential series

14.7 TAYLOR'S THEOREM

If a function $f(z)$ is analytic at all points inside a circle C, with its centre at the point a and radius R, then at each point z inside C.

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^n(a)}{n!}(z-a)^n + \ldots$$

Proof. Take any point z inside C. Draw a circle C_1 with centre a, enclosing the point z. Let w be a point on circle C_1.



$$= \frac{1}{w-z} = \frac{1}{w-a+a-z} = \frac{1}{w-a-(z-a)}$$

$$= \frac{1}{(w-a)}\frac{1}{\left(1-\frac{z-a}{w-a}\right)}$$

$$= \frac{1}{w-a}\left(1-\frac{z-a}{w-a}\right)^{-1}$$

Apply Binomial theorem

$$\frac{1}{w-z} = \frac{1}{w-a}\left[1+\frac{z-a}{w-a}+\left(\frac{z-a}{w-a}\right)^2+\ldots+\left(\frac{z-a}{w-a}\right)^n+\ldots\right]$$

$$\frac{1}{w-z} = \frac{1}{w-a}+\frac{z-a}{(w-a)^2}+\frac{(z-a)^2}{(w-a)^3}+\ldots+\frac{(z-a)^n}{(w-a)^{n+1}}+\ldots \quad \ldots(1)$$

As $\quad |z-a| < |w-a| \Rightarrow \quad \frac{|z-a|}{|w-a|} < 1,$

So the series converges uniformly. Hence the series is integrable.
Multiply (1) by $f(w)$.

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-a} + (z-a)\frac{f(w)}{(w-a)^2} + (z-a)^2\frac{f(w)}{(w-a)^3} + \ldots + (z-a)^n\frac{f(w)}{(w-a)^{n+1}} + ,\ldots$$

On integrating w.r.t. 'w', we get

$$\int_{c_1}\frac{f(w)}{w-z}dw = \int_{c_1}\frac{f(w)}{w-a}dw + (z-a)\int_{c_1}\frac{f(w)}{(w-a)^2}dw + (z-a)^2\int_{c_1}\frac{f(w)}{(w-a)^3}dw +$$

$$\ldots + (z-a)^n\int_{c_1}\frac{f(w)}{(w-a)^{n+1}}dw + \ldots \quad \ldots(2)$$

We know that

$$\int_{c_1}\frac{f(w)}{w-z}dz = 2\pi i f(z) \text{ and } \int_{c_1}\frac{f(w)}{w-a}dw = 2\pi i f(a)$$

$$\int_{c_1}\frac{f(w)}{(w-a)^2}dw = 2\pi i f'(a) \text{ and so on.}$$

Substituting ... (2), we get Taylor's series as given below

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots + \frac{f^n(a)(z-a)^n}{n!} + \ldots \quad \ldots(3)$$ **Proved.**

Corollary 1. Putting $z = a + h$ in (3), we get

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots\ldots + \frac{h^n}{n!}f^n(a) + \ldots$$

Corollary 2. If $a = 0$, the series (3) becomes

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!}f''(0) + \ldots\ldots + \frac{z^n}{n!}f^n(0) + \ldots$$

This series is called **Maclaurin's series.**

Example 5. *Expand e^z about a.*

Solution. Here, we have

$$f(z) = e^z \qquad \Rightarrow \qquad f(a) = e^a$$
$$f'(z) = e^z \qquad \Rightarrow \qquad f'(a) = e^a$$
$$\Rightarrow \quad f''(z) = e^z \qquad \Rightarrow \qquad f''(a) = e^a$$

....

....

$$f^n(z) = e^z \qquad \Rightarrow \qquad f^n(a) = e^a$$

Taylor's series of $f(z)$ is

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots\ldots\ldots$$

$$\Rightarrow \quad e^z = e^a + \frac{(z-a)}{1!}e^a + \frac{(z-a)^2}{2!}e^a + \ldots\ldots\ldots$$ **Ans.**

Example 6. *Expand the function*

$$f(z) = \frac{1}{z}$$

about $z = 2$ in Taylor's series. Obtain its radius of convergence.

Solution. Here, we have,

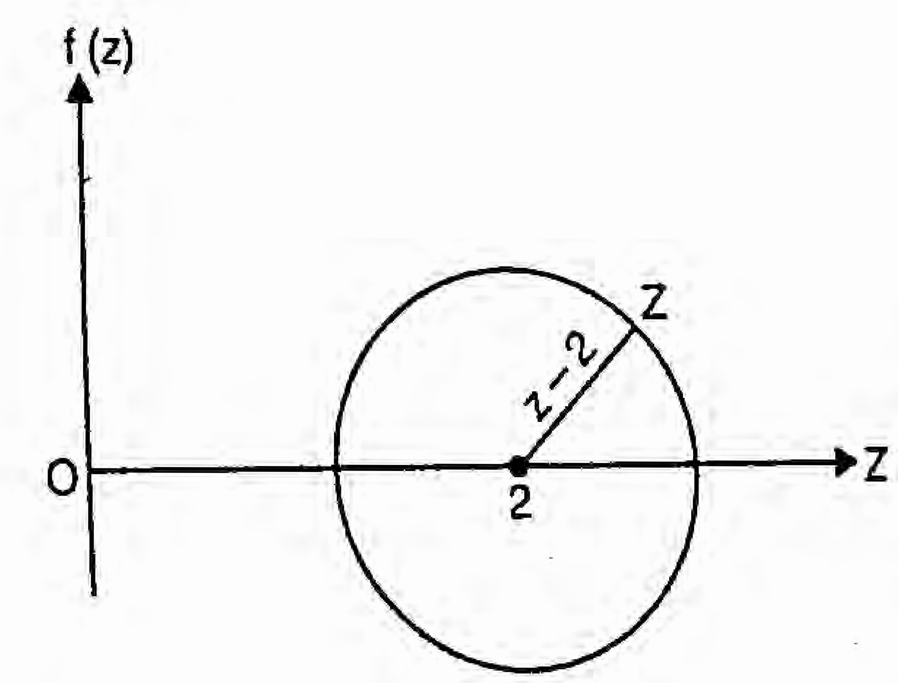
$$f(z) = \frac{1}{z}$$

$$\Rightarrow \quad f'(z) = -\frac{1}{z^2}$$

$$\Rightarrow \quad f''(z) = \frac{2}{z^3}$$

....

$$f^n(z) = (-1)^n\frac{n!}{z^{n+1}}$$



By Taylor's series

$$f(z) = f(2) + (z-2)f'(2) + \frac{(z-2)^2}{2!}f''(2) + \ldots\ldots\ldots + \frac{(z-2)^n}{n!}f^n(2) + \ldots\ldots$$

$$= \frac{1}{2} + (z-2)\left(-\frac{1}{2^2}\right) + \frac{(z-2)^2}{2!}\left(\frac{2}{2^3}\right) + \ldots\ldots \frac{(z-2)^n}{n!}(-1)^n \frac{n!}{2^{n+1}} + \ldots$$

$$= \frac{1}{2} - \frac{1}{4}(z-2) + \frac{1}{8}(z-2)^2 - \ldots\ldots + \frac{(-1)^n}{2^{n+1}}(z-2)^n + \ldots\ldots$$

Ans.

Alternative.

We can expand the given function by Binomial expansion.

$$\frac{1}{z} = \frac{1}{2+z-2} = \frac{1}{2}\left[\frac{1}{1+\frac{z-2}{2}}\right] = \frac{1}{2}\left[\left(1+\frac{z-2}{2}\right)^{-1}\right], \quad \left|\frac{z-2}{2}\right| < 1$$

$$= \frac{1}{2}\left[1 - \frac{z-2}{2} + \frac{(-1)(-2)}{2!}\left(\frac{z-2}{2}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{z-2}{2}\right)^3 + \ldots\right]$$

$$= \frac{1}{2} - \frac{z-2}{4} + \frac{1}{8}(z-2)^2 - \frac{1}{16}(z-2)^3 + \ldots\ldots$$

Ans.

Radius of convergence $\frac{1}{R} = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{2^{n+2}} \cdot \frac{2^{n+1}}{(-1)^n}\right| = \frac{1}{2}$

$$\Rightarrow \quad R = 2$$

Ans.

Example 7. *Expand $f(z) = \cosh z$ about πi.*

Solution. Here, we have

$$f(z) = \cosh z = \frac{e^z + e^{-z}}{2} \Rightarrow f(\pi i) = \cosh(\pi i)$$

$$f'(z) = \sinh z \quad \Rightarrow \quad f'(\pi i) = \sinh(\pi i) = i \sin \pi = 0$$

$$f''(z) = \cosh z \quad \Rightarrow \quad f''(\pi i) = \cosh(\pi i)$$

$$f'''(z) = \sinh z \quad \Rightarrow \quad f'''(\pi i) = \sinh(\pi i)$$

By Taylor's series

$$f(z) = f(\pi i) + (z-\pi i) f'(\pi i) + \frac{(z-\pi i)^2}{2!} f''(\pi i) + \frac{(z-\pi i)^3}{3!} f'''(\pi i) + \ldots$$

$$= \cosh \pi i + (z-\pi i)\sinh \pi i + \frac{(z-\pi i)^2}{2!}\cosh(\pi i) + \frac{(z-\pi i)^3}{3!}\sinh(\pi i) + \ldots\ldots$$

$$f(z) = -1 + \frac{(2-\pi i)^2}{a^{21}} - \frac{(z-\pi i)^4}{41} \ldots\ldots$$

Ans.

Example 8. *Expand $f(z) = \frac{a}{bz+c}$ about z_0.*

Solution. Here, we have

$$f(z) = \frac{a}{bz+c}$$

$$= \frac{a}{bz - bz_0 + bz_0 + c} = \frac{a}{b(z-z_0) + bz_0 + c}$$

$$= \frac{a}{(bz_0+c)\left[1 + \left(\frac{b(z-z_0)}{bz_0+c}\right)\right]}$$

$$= \frac{a}{bz_0+c}\left[1 + \frac{b(z-z_0)}{bz_0+c}\right]^{-1}$$

$$= \frac{a}{bz_0+c}\left[1 - \frac{b(z-z_0)}{bz_0+c} + \frac{(-1)(-2)}{2!}\left(\frac{b(z-z_0)}{bz_0+c}\right)^2 + \frac{(-1)(-2)(-3)}{3!}\left(\frac{b(z-z_0)}{bz_0+c}\right)^3 + \ldots\ldots\right]$$ (Binomial series)

$$= \frac{a}{bz_0+c}\left[1 - \frac{b(z-z_0)}{bz_0+c} + \left(\frac{b(z-z_0)}{bz_0+c}\right)^2 - \left(\frac{b(z-z_0)}{bz_0+c}\right)^3 + \ldots\ldots\right]$$

$$= \frac{a}{bz_0+c}\left[1 - \frac{b}{bz_0+c}(z-z_0) + \left(\frac{b}{bz_0+c}\right)^2 (z-z_0)^2 - \left(\frac{b}{bz_0+c}\right)^3 (z-z_0)^3 + \ldots\right]$$

Radius of curvature

$$\frac{1}{R} = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right)$$

$$= \lim_{n\to\infty}\frac{\left(\frac{b}{bz_0+c}\right)^{n+1}}{\left(\frac{b}{bz_0+c}\right)^n} = \lim_{n\to\infty}\frac{b}{bz_0+c} = \frac{b}{bz_0+c}$$

$$\Rightarrow \quad R = \frac{bz_0+c}{b} = z_0 + \frac{c}{b}$$

Ans.

Example 9. *Show that :*

$$\log z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} + \ldots\ldots$$

Solution. Let $f(2) = \log 2$, $\quad f(1) = \log 1 = 0$

$$f'(z) = \frac{1}{z}, \qquad f'(1) = \frac{1}{1} = 1$$

$$f''(z) = -\frac{1}{z^2}, \qquad f''(1) = -1$$

$$f'''(z) = \frac{2\times 1}{z^3}, \qquad f'''(1) = 2$$

$$f^{iv}(z) = -3\times 2\times 1\times\frac{1}{z^4}, \quad f^{iv}(1) = -3!$$

By Taylors series

$$f(z) = f(a) + f'(a).(z-a) + \frac{f''(a)(z-a)^2}{2!} + \frac{f'''(a)}{3!}(z-a)^3 + ...$$

$$f(z) = \log z = \log\,(1 + \overline{z-1})$$

On substituting the values of $f(1), f'(1), f''(1)$ etc., we get

$$\log z = 0 + 1(z-1) - \frac{1}{2!}(z-1)^2 + \frac{2}{3!}(z-1)^3 - \frac{3!}{4!}(z-1)^4 + ...$$

$$\Rightarrow \quad \log z = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{4}(z-1)^4 + ...$$

Proved.

Example 10. *Obtain the terms upto z^3 in the Taylor's series expansion of*

$$f(z) = \frac{(z^2 + \sin^2 z)}{1 - \cos z}$$

about the point $z = 0$. Find its radius of convergence.

Solution. The function $f(z)$ is not analytic when $1 - \cos z = 0$, that is when $z = 2n\pi$, n is any integer.

At $z = 0$, $f(z)$ has a limiting value 4. We have

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...$$

$$\sin^2 z = z^2 - \frac{z^4}{3} + \frac{2}{45}z^6 + ...$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + ...$$

Therefore,

$$f(z) = \frac{z^2 + \sin^2 z}{1 - \cos z} = \frac{z^2 + \left[z^2 - \frac{z^4}{3} + \frac{2z^6}{45} - ....\right]}{1 - \left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...\right]}$$

$$= \frac{z^2\left[2 - \frac{z^2}{3} + \frac{2z^4}{45} - ....\right]}{\frac{z^2}{2}\left[1 - \frac{z^2}{12} + \frac{z^4}{360} - ....\right]}$$ (Binomial series)

$$= 2\left[2 - \frac{z^2}{3} + \frac{2z^4}{45} - ....\right]\left[1 - \left(\frac{z^2}{12} - \frac{z^4}{360} + ...\right)\right]^{-1}$$

$$= 2\left[2 - \frac{z^2}{3} + \frac{2z^4}{45} - ....\right]\left[1 + \frac{z^2}{12} - \frac{3}{720}z^4 - ....\right]$$

$$= 2\left[2 + \frac{z^2}{6} - \frac{1}{120}z^4 - \frac{z^2}{3} - \frac{z^4}{36} + \frac{2}{45}z^4 + ...\right]$$

$$= 2\left[2 + \frac{z^2}{6} - \frac{z^2}{3}\right] = 2\left[2 - \frac{z^2}{6} + 0(z^4)\right] = 4 - \frac{z^2}{3} + 0(z^4)$$

The distance between $z = 0$ and the nearest points $z = \pm 2\pi$. At which the function $f(z)$ is not analytic is 2π.

Therefore, $R = 2\pi$

Example 11. *Expand $\frac{1}{z^2 - 3z + 2}$ in the region*

(a) $|z| < 1$ (b) $|z| > 2$. (c) $1 < |z| < 2$

(R.G.P.V., Bhopal, III Semester, Dec. 2005)

Ans.

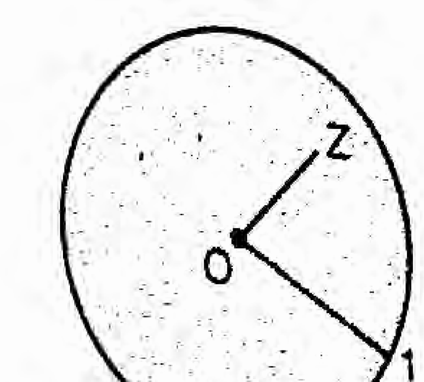
Solution. Here, $f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

(a) If $|z| < 1$

Taking common, bigger term out of $|z|$ and 2, here 2 is bigger than $|z|$. So we take 2 common.

$$f(z) = \frac{1}{-2\left(1 - \frac{z}{2}\right)} + \frac{1}{1-z}$$

$$= -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + (1-z)^{-1}$$



$$= -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + ...\right) + (1 + z + z^2 + z^3 + ...)$$

[By Binomial theorem]

$$= \frac{1}{2} + \frac{3z}{4} + \frac{7}{8}z^2 + \frac{15}{16}z^3 + ...$$

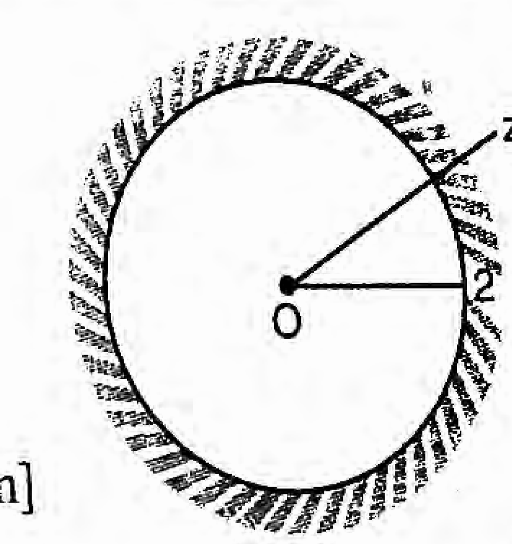
Which is the required expansion.

(b) If $|z| > 2$

We have, $f(z) = \frac{1}{z-2} - \frac{1}{z-1}$

Taking common, bigger term out of $|z|$ and 2, here z is bigger than 2. So we take $|z|$ common.

$$f(z) = \frac{1}{z\left(1 - \frac{2}{z}\right)} - \frac{1}{z\left(1 - \frac{1}{z}\right)} = \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$



$$= \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + ...\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ...\right)$$

[By Binomial theorem]

$$= z^{-2} + 3z^{-3} + 7z^{-4} + ...$$

Ans.

Which is the required expansion.

(c) For the expansion of the given function by Binomial expansion is valid where $1 < |z| < 2$.

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= -\frac{1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}} \qquad [1<|z|<2]$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}......\right] - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+....\right)$$

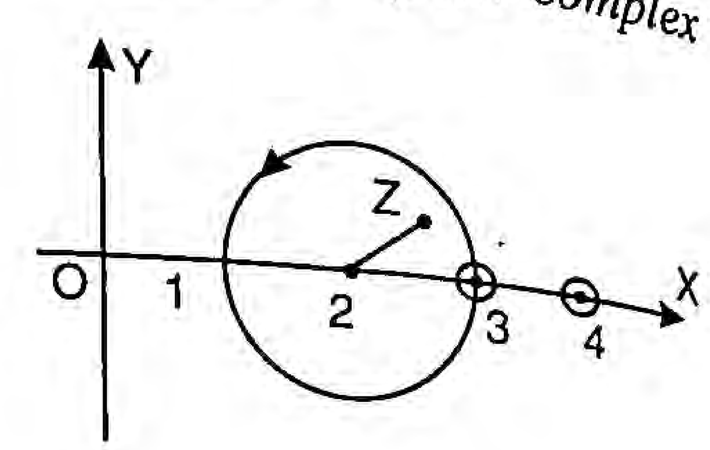
$$= -\frac{1}{2}-\frac{z}{4}-\frac{z^2}{8}-\frac{z^3}{16}-..........-\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{z^4}......$$

Which is the required expansion. **Ans.**

Example 12. *Find the first four terms of the Taylor's series expansion of the complex variable function*

$$f(z)=\frac{z+1}{(z-3)(z-4)}$$

about z = 2. Find the region of convergence.



Solution. $f(z)=\dfrac{z+1}{(z-3)(z-4)}$

If centre of a circle is at $z = 2$, then the distances of the singularities $z = 3$ and $z = 4$ from the centre are 1 and 2.

Hence if a circle is drawn with centre $z = 2$ and radius 1, then within the circle $|z-2| = 1$, the given function $f(z)$ is analytic, hence it can be expanded in a Taylor's series within the circle $|z-2| = 1$, which is therefore the circle of convergence.

$$f(z)=\frac{z+1}{(z-3)(z-4)}=\frac{-4}{z-3}+\frac{5}{z-4} \qquad \text{(By Partial fraction method)}$$

$$=\frac{-4}{(z-2)-1}+\frac{5}{(z-2)-2}=4[1-(z-2)]^{-1}-\frac{5}{2}\left[1-\frac{z-2}{2}\right]^{-1} \qquad [|z-2|<1]$$

$$=4[1+(z-2)+(z-2)^2+(z-2)^3+...]-\frac{5}{2}\left[1+\frac{z-2}{2}+\frac{(z-2)^2}{4}+\frac{(z-2)^3}{8}+...\right]$$

$$=\left(4-\frac{5}{2}\right)+\left(4-\frac{5}{4}\right)(z-2)+\left(4-\frac{5}{8}\right)(z-2)^2+\left(4-\frac{5}{16}\right)(z-2)^3+...$$

$$=\frac{3}{2}+\frac{11}{4}\ (z-2)+\frac{27}{8}\ (z-2)^2+\frac{59}{16}\ (z-2)^3+...$$ **Ans.**

Alternative method. In obtaining the Taylor series we evaluate the coefficients by contour integration.

$$f(z)=\frac{z+1}{(z-3)(z-4)}, \qquad f(2)=\frac{2+1}{(2-3)(2-4)}=\frac{3}{2}.$$

To make the differentiation easier let us convert the given fraction into partial fractions

$$f(z)=\frac{-4}{z-3}+\frac{5}{z-4}$$

$$f'(z)=\frac{4}{(z-3)^2}-\frac{5}{(z-4)^2}, \qquad f'(2)=\frac{4}{(2-3)^2}-\frac{5}{(2-4)^2}=\frac{11}{4}$$

$$f''(z)=\frac{-8}{(z-3)^3}+\frac{10}{(z-4)^3}, \qquad f''(2)=\frac{-8}{(2-3)^3}+\frac{10}{(2-4)^3}=\frac{27}{4}$$

$$f'''(z)=\frac{24}{(z-3)^4}-\frac{30}{(z-4)^4}, \qquad f'''(2)=\frac{24}{(2-3)^4}-\frac{30}{(2-4)^4}=\frac{177}{8}$$

Taylor series is $f(z)=f(a)+(z-a)f'(a)+\dfrac{(z-a)^2}{2!}f''(a)+\dfrac{(z-a)^3}{3!}f'''(a)+...$

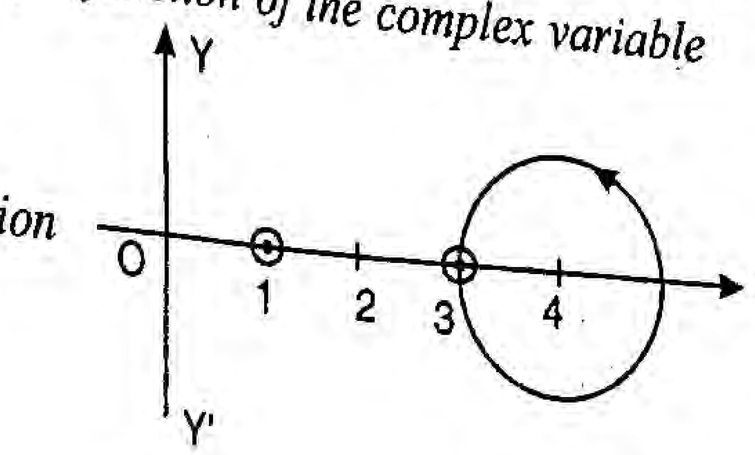
$$\frac{z+1}{(z-3)(z-4)}=\frac{3}{2}+(z-2)\frac{11}{4}+\frac{(z-2)^2}{2!}\left(\frac{27}{4}\right)+\frac{(z-2)^3}{3!}\frac{177}{8}+...$$

$$=\frac{3}{2}+(z-2)\frac{11}{4}+(z-2)^2.\frac{27}{8}+(z-2)^3.\frac{59}{16}+...$$ **Ans.**

Example 13. *Find the Taylor series expansion of a function of the complex variable*

$$f(z)=\frac{1}{(z-1)(z-3)}$$

about the point z = 4. Find its region of convergence.



Solution. $f(z)=\dfrac{1}{(z-1)(z-3)}.$

If the centre of a circle is at $z = 4$, then the distances of the singularities $z = 1$ and $z = 3$ from the centre are 3 and 1 respectively. Hence, if a circle is drawn with centre at $z = 4$ and radius 1, then, within the circle $|z-4| = 1$ the given function $f(z)$ is analytic. Hence, it can be expanded in Taylor series within the circle $|z-4| = 1$, which is therefore the circle of convergence.

$$f(z)=\frac{1}{(z-1)(z-3)}=\frac{1}{2}\left(\frac{1}{z-3}-\frac{1}{z-1}\right)$$

$$=\frac{1}{2}\left[\frac{1}{z-4+1}-\frac{1}{(z-4)+3}\right]$$

Taking common, bigger term out of $|z-4|$ and 3, here 3 is bigger than $|z-4|$, so we take 3 common.

$$f(z)=\frac{1}{2}\left[\frac{1}{1+(z-4)}-\frac{1}{3}\frac{1}{1+\frac{z-4}{3}}\right]$$

$$\Rightarrow \quad f(z)=\frac{1}{2}[1+(z-4)]^{-1}-\frac{1}{6}\left[1+\frac{z-4}{3}\right]^{-1}$$

$$=\frac{1}{2}[1-(z-4)+(z-4)^2-(z-4)^3+...]-\frac{1}{6}\left[1-\frac{z-4}{3}+\frac{(z-4)^2}{9}-\frac{(z-4)^3}{27}+...\right]$$

$$=\left(\frac{1}{2}-\frac{1}{6}\right)+\left(-\frac{1}{2}+\frac{1}{18}\right)(z-4)+\left(\frac{1}{2}-\frac{1}{54}\right)(z-4)^2+\left(-\frac{1}{2}+\frac{1}{162}\right)(z-4)^3+...$$

$$f(z)=\frac{1}{3}-\frac{4}{9}(z-4)+\frac{13}{27}(z-4)^2-\frac{40}{81}(z-4)^3+\dots$$

Which is the required Taylor's Series expansion of the given function of the complex variable.

Ans.

Alternative method. In obtaining the expansion of the given function by Taylor's series we evaluate the coefficients by contour integration.

$$f(z)=\frac{1}{(z-1)(z-3)},\qquad f(4)=\frac{1}{(4-1)(4-3)}=\frac{1}{3}$$

Let us convert the given function into partial fractions to make differentiation easier.

$$f(z)=\frac{1}{2}\left(\frac{1}{z-3}-\frac{1}{z-1}\right)$$

$$f'(z)=\frac{1}{2}\left[-\frac{1}{(z-3)^2}+\frac{1}{(z-1)^2}\right],\qquad f'(4)=\frac{1}{2}\left[-\frac{1}{(4-3)^2}+\frac{1}{(4-1)^2}\right]=-\frac{4}{9}$$

$$f''(z)=\frac{1}{2}\left[\frac{2}{(z-3)^3}-\frac{2}{(z-1)^3}\right],\qquad f''(4)=\frac{1}{2}\left[\frac{2}{(4-3)^3}-\frac{2}{(4-1)^3}\right]=\frac{26}{27}$$

$$f'''(z)=\frac{1}{2}\left[-\frac{6}{(z-3)^4}+\frac{6}{(z-1)^4}\right],\qquad f'''(4)=\frac{1}{2}\left[-\frac{6}{(4-3)^4}+\frac{6}{(4-1)^4}\right]=-\frac{80}{27}$$

Taylor's series is

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\frac{(z-a)^3}{3!}f'''(a)+\dots$$

$$\frac{1}{(z-1)(z-3)}=\frac{1}{3}+(z-4)\left(-\frac{4}{9}\right)+\frac{(z-4)^2}{2!}\left(\frac{26}{27}\right)+\frac{(z-4)^3}{3!}\left(-\frac{80}{27}\right)+\dots$$

$$=\frac{1}{3}-\frac{4}{9}(z-4)+(z-4)^2\cdot\frac{13}{27}-(z-4)^3\frac{40}{81}+\dots$$

Which is the required Taylor's Series expansion of the given function of the complex variable.

Ans.

Example 14. *Find the first three terms of the Taylor series expansion of* $f(z)=\frac{1}{z^2+4}$ *about* $z=-i$. *Find the region of convergence.*

Solution. $f(z)=\frac{1}{z^2+4}$

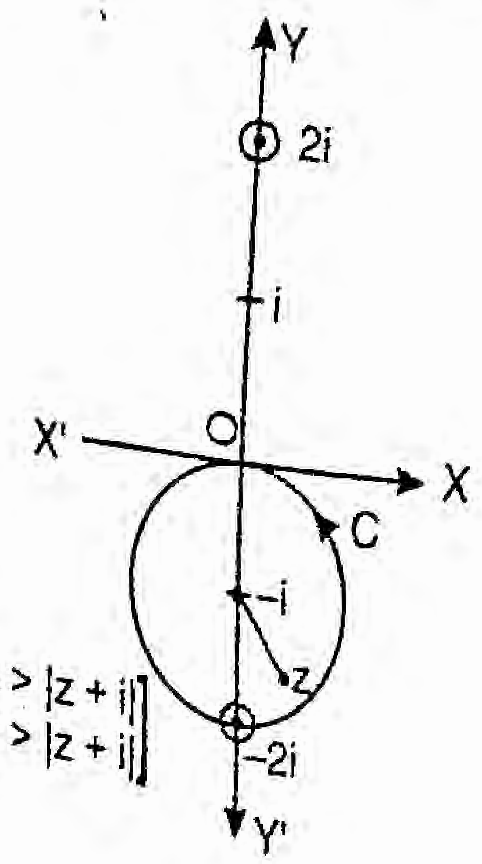
Poles are given by $z^2+4=0$, $\Rightarrow$ $z^2=-4$, $\Rightarrow$ $z=\pm 2i$

If the centre of a circle is $z=-i$, then the distances of the singularities $z=2i$ and $z=-2i$ from the centre are 3 and 1. Hence if a circle of radius 1 is drawn with centre at $z=-i$, then within the circle $|z+i|=1$, the given function $f(z)$ is analytic. Thus the function can be expanded in Taylor series within the circle $|z+i|=1$, which is therefore the circle of convergence.

$$f(z)=\frac{1}{z^2+4}=\frac{1}{(z+2i)(z-2i)}$$

$$=\frac{1}{4i}\left[\frac{1}{z-2i}-\frac{1}{z+2i}\right]$$

$$=\frac{1}{4i}\left[\frac{1}{(z+i)-3i}-\frac{1}{(z+i)+i}\right]$$



$$\begin{bmatrix}3>|z+i|\\1>|z+i|\end{bmatrix}$$

$$=\frac{1}{4i}\left[-\frac{1}{3i}\frac{1}{1-\frac{z+i}{3i}}-\frac{1}{i}\frac{1}{1+\frac{z+i}{i}}\right]=\frac{1}{4}\left[\frac{1}{3}\frac{1}{1-\frac{z+i}{3i}}+\frac{1}{1+\frac{z+i}{i}}\right]$$

$$=\frac{1}{4}\left[\frac{1}{3}\frac{1}{1+\frac{i}{3}(z+i)}+\frac{1}{1-i(z+i)}\right]=\frac{1}{4}\left[\frac{1}{3}\left\{1+\frac{i}{3}(z+i)\right\}^{-1}+\{1-i(z+i)\}^{-1}\right]$$

$$=\frac{1}{4}\left[\frac{1}{3}\left\{1-\frac{i}{3}(z+i)+\frac{(-1)(-2)}{2!}\left(\frac{i}{3}\right)^2(z+i)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{i}{3}\right)^3(z+i)^3+\dots\right\}\right.$$
$$\left.+\left\{1+i(z+i)+\frac{(-1)(-2)}{2!}\cdot(-i)^2(z+i)^2+\frac{(-1)(-2)(-3)}{3!}(-i)^3(z+i)^3+\dots\right\}\right]$$

$$=\frac{1}{4}\left[\frac{1}{3}-\frac{i}{9}(z+i)-\frac{1}{27}(z+i)^2+\frac{i}{81}(z+i)^3+\dots+1+i(z+i)-(z+i)^2-i(z+i)^3+\dots\right]$$

$$=\frac{1}{4}\left[\frac{4}{3}+\frac{8i}{9}(z+i)-\frac{28}{27}(z+i)^2-\frac{80}{81}i(z+i)^3+\dots\right]$$

$$\therefore\ f(z)=\frac{1}{3}+\frac{2i}{9}(z+i)-\frac{7}{27}(z+i)^2-\frac{20}{81}i(z+i)^3+\dots$$

Ans.

Alternative method

$$f(z)=\frac{1}{z^2+4}\qquad\dots(1)$$

By Taylor expansion $f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\frac{(z-a)^3}{3!}f'''(a)+\dots$ ...(2)

Putting $a=-i$ in (2), we get

$$f(z)=f(-i)+(z+i)f'(-i)+\frac{(z+i)^2}{2!}f''(-i)+\dots$$

$$f(z)=\frac{1}{z^2+4}$$

$$\Rightarrow\qquad f(-i)=\frac{1}{(-i)^2+4}=\frac{1}{-1+4}=\frac{1}{3}$$

$$f'(z)=\frac{-2z}{(z^2+4)^2}.$$

6. $\log\left(\frac{1+z}{1-z}\right)$ about $z=0$ **Ans.** $\sum_{n=0}^{\infty} \frac{2z^{2n+1}}{2n+1}$, $|z|<1$

7. $\frac{1}{z^2+(1+2i)z+2i}$ about $z=0$. **Ans.** $\frac{1}{(1-2i)}\left[\sum_{n=0}^{\infty}\left\{\left(\frac{1}{2i}\right)^{n+1}-1\right\}(-1)^n z^n\right]$

8. $(z+i)^{\frac{1}{2}}$, about $z=0$ **Ans.** $\frac{1+i}{\sqrt{2}}\left[\sum_{n=0}^{\infty} {}^{\frac{1}{2}}C_r\left(\frac{z}{i}\right)^r\right]$, $R=1$

9. $\frac{1}{(z^2-1)(z^2-2)}$ about $z=0$. **Ans.** $\sum_{n=0}^{\infty}\left[1-\frac{1}{2^{n+1}}\right]z^{2n}$, $R=1$.

10. $\tan z$ about $z=0$ **Ans.** $z+\frac{z^3}{3}+\frac{2z^5}{15}+\ldots$

11. $z\cot z$ about $z=0$ **Ans.** $1-\frac{z^2}{3}-\frac{z^4}{45}+\ldots$

12. $\frac{e^z}{1+e^z}$ about $z=0$ **Ans.** $\frac{1}{2}+\frac{z}{4}-\frac{z^3}{48}$

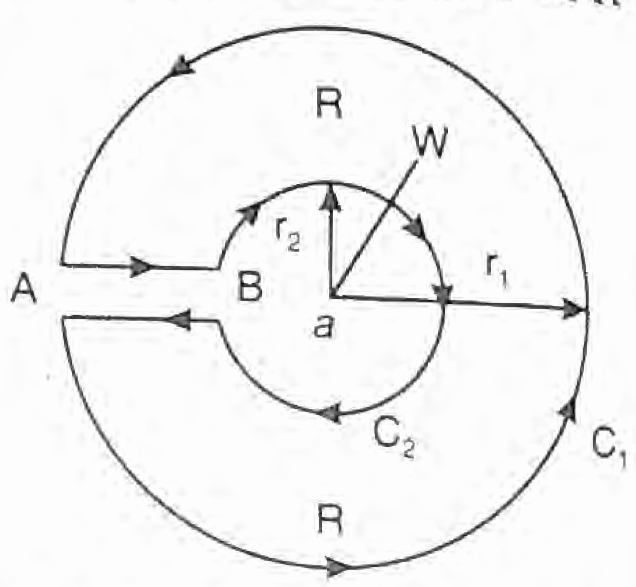
14.8 LAURENT'S THEOREM (U.P. III Semester Dec. 2009)

If we are required to expand $f(z)$ about a point where $f(z)$ is not analytic, then it is expanded by Laurent's Series and not by Taylor's Series.

Statement. If $f(z)$ is analytic on c_1 and c_2, and the annular region R bounded by the two concentric circles c_1 and c_2 of radii r_1 and r_2 $(r_2<r_1)$ and with centre at a, then for all z in R

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+\ldots+\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\ldots$$

where
$$a_n=\frac{1}{2\pi i}\int_{c_1}\frac{f(w)}{(w-a)^{n+1}}dw$$
$$b_n=\frac{1}{2\pi i}\int_{c_2}\frac{f(w)}{(w-a)^{-n+1}}dw$$



Proof. By introducing a cross cut AB, multi-connected region R is converted to a simply connected region. Now $f(z)$ is analytic in this region.

Now by Cauchy's integral formula

$$f(z)=\frac{1}{2\pi i}\int_{c_1}\frac{f(w)dw}{w-z}+\frac{1}{2\pi i}\int_{AB}\frac{f(w)}{w-z}dw-\frac{1}{2\pi i}\int_{c_2}\frac{f(w)dw}{w-z}+\frac{1}{2\pi i}\int_{BA}\frac{f(w)}{w-z}dw$$

Integral along c_2 is clockwise, so it is negative. Integrals along AB and BA cancel.

$$f(z)=\frac{1}{2\pi i}\int_{c_1}\frac{f(w)dw}{w-z}-\frac{1}{2\pi i}\int_{c_2}\frac{f(w)dw}{w-z} \quad \ldots(1)$$

For the first integral, $\frac{f(w)}{w-z}$ can be expanded exactly as in Taylor's series as z lies on c_1.

$$\left(\frac{|z-a|}{|w-a|}\le 1 \quad \because w \text{ lies on } c_1\right)$$

$$\frac{1}{2\pi i}\int_{c_1}\frac{f(w)dw}{w-z}=\frac{1}{2\pi i}\int_{c_1}\frac{f(w)}{w-a}dw+\frac{(z-a)}{2\pi i}\int_{c_1}\frac{f(w)}{(w-a)^2}dw+\frac{(z-a)^2}{2\pi i}\int_{c_1}\frac{f(w)}{(w-a)^3}dw+\ldots$$

$$=a_0+a_1(z-a)+a_2(z-a)^2+\ldots \quad \ldots(2) \quad \left[a_n=\frac{1}{2\pi i}\int_{c_1}\frac{f(w)}{(w-a)^{n+1}}dw\right]$$

In the second integral, z lies on c_2. Therefore

$$|w-a|<|z-a| \quad \text{or} \quad \frac{|w-a|}{|z-a|}<1$$

so here
$$\frac{1}{w-z}=\frac{1}{w-a+a-z}=\frac{1}{(w-a)-(z-a)}$$
$$=-\frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)}=-\frac{1}{z-a}\left(1-\frac{w-a}{z-a}\right)^{-1}$$
$$=-\frac{1}{z-a}\left[1+\frac{w-a}{z-a}+\left(\frac{w-a}{z-a}\right)^2+\ldots+\left(\frac{w-a}{z-a}\right)^{n+1}+\ldots\right]$$

Multiplying by $-\frac{f(w)}{2\pi i}$, we get

$$-\frac{1}{2\pi i}\frac{f(w)}{w-z}=\frac{1}{2\pi i}\frac{f(w)}{z-a}+\frac{1}{2\pi i}\frac{(w-a)}{(z-a)^2}f(w)+\frac{1}{2\pi i}\frac{(w-a)^2}{(z-a)^3}f(w)+\ldots$$
$$=\left(\frac{1}{z-a}\right)\frac{1}{2\pi i}f(w)+\frac{1}{(z-a)^2}\frac{1}{2\pi i}\frac{f(w)}{(w-a)^{-1}}+\frac{1}{(z-a)^3}\frac{1}{2\pi i}\frac{f(w)}{(w-a)^{-2}}+\ldots$$

Integrating, we have

$$-\frac{1}{2\pi i}\int_{c_2}\frac{f(w)}{w-z}dw=\left(\frac{1}{z-a}\right)\frac{1}{2\pi i}\int_{c_2}f(w)dw+\frac{1}{(z-a)^2}\frac{1}{2\pi i}\int_{c_2}\frac{f(w)dw}{(w-a)^{-1}}$$
$$+\frac{1}{(z-a)^3}\frac{1}{2\pi i}\int_{c_2}\frac{f(w)dw}{(w-a)^{-2}}+\ldots$$
$$=\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\frac{b_3}{(z-a)^3}+\ldots \quad \ldots(3) \quad \left[b_n=\frac{1}{2\pi i}\int_{c_2}\frac{f(w)dw}{(w-a)^{-n+1}}\right]$$

Substituting the values of both integrals from (2) and (3) in (1), we get

$$f(z)=a_0+a_1(z-a)+a_2(z-a)^2+b_1(z-a)^{-1}+b_2(z-a)^{-2}+\ldots$$

$$\Rightarrow \quad f(z)=\sum_{n=0}^{n=\infty}a_n(z-a)^n+\sum_{n=1}^{n=\infty}\frac{b_n}{(z-a)^n}$$ **Proved.**

Note. To expand a function by Laurent Theorem is cumbersome. By Binomial theorem, the expansion of a function can be done easily.

Example 22. *Expand* $f(z)=\frac{z}{(z-1)(2-z)}$ *in Laurent series valid for*

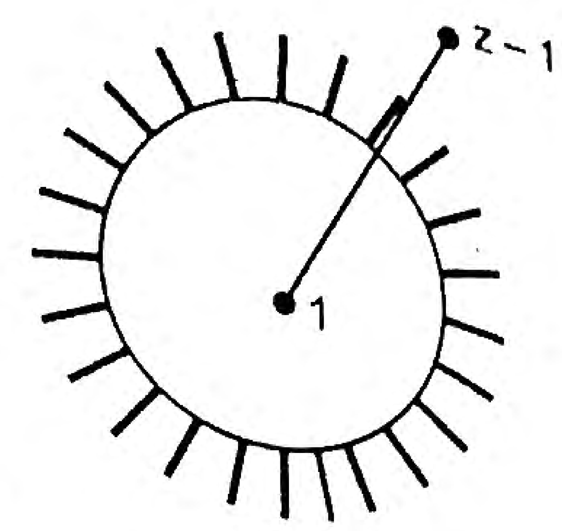
(i) $|z-1|>1$ *and* *(ii)* $0<|z-2|<1$.

(G.B.T.U., III Semester, Dec. 2012, 2011, 2010)

Solution: $f(z)=\frac{z}{(z-1)(2-z)}=\frac{1}{z-1}+\frac{2}{2-z}$

(i) $|z-1|>1 \Rightarrow \frac{1}{|z-1|}<1$

$$f(z) = \frac{1}{z-1} - \frac{2}{(z-1)-1} = \frac{1}{z-1} - \frac{2}{z-1}\left(\frac{1}{1-\frac{1}{z-1}}\right)$$



$$= \frac{1}{z-1} - \frac{2}{z-1}\left(1-\frac{1}{z-1}\right)^{-1}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left(1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+\ldots\right)$$

which is required Laurent expansion.

(*ii*) $0 < |z-2| < 1$.

$$f(z) = \frac{2}{2-z} + \frac{1}{(z-2)+1}$$

$$= \frac{2}{2-z} + \frac{1}{1+(z-2)}$$

$$= \frac{2}{2-z} + [1+(z-2)]^{-1}$$

$$= \frac{2}{2-z} + [1-(z-2)+(z-2)^2-(z-2)^3+\ldots]$$

which is required Laurent expansion. **Ans.**

Example 23. *Obtain the Taylor or Laurent series which represents the function*

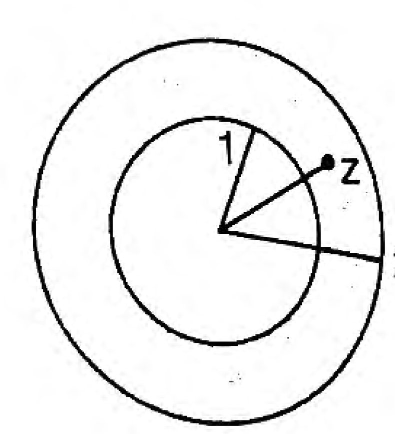
$$f(z) = \frac{1}{(1+z^2)(z+2)} \quad \textit{when}$$

(*i*) $1 < |z| < 2$; (*ii*) $|z| > 2$

Solution. $f(z) = \dfrac{1}{(1+z^2)(z+2)} = \dfrac{-\frac{z}{5}+\frac{2}{5}}{1+z^2} + \dfrac{\frac{1}{5}}{z+2} = -\dfrac{1}{5}\dfrac{z-2}{1+z^2} + \dfrac{1}{5}\dfrac{1}{z+2}$ $[1<|z|<2]$

(*i*) In first expression $1 < |z|$ and in second expression $|z| < 2$

$$f(z) = -\frac{1}{5}\frac{1}{z^2}\frac{z-2}{1+\frac{1}{z^2}} + \frac{1}{5}\frac{1}{2}\frac{1}{1+\frac{z}{2}}$$



$$f(z) = -\frac{1}{5z^2}(z-2)\left(1+\frac{1}{z^2}\right)^{-1} + \frac{1}{10}\left(1+\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{5}\left(\frac{1}{z}-\frac{2}{z^2}\right)\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\ldots\right) + \frac{1}{10}\left[1-\frac{z}{2}+\frac{z^2}{4}-\frac{z^3}{8}+\ldots\right]$$

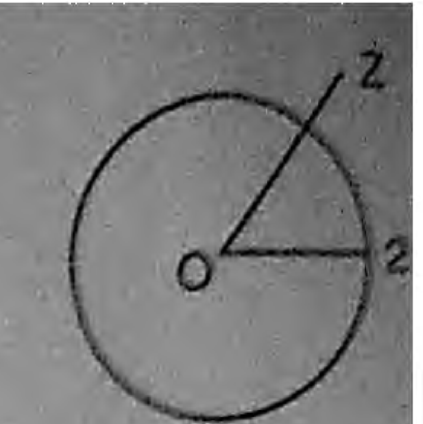
$$= \frac{1}{5}\left[-\frac{1}{z}+\frac{1}{z^3}-\frac{1}{z^5}+\frac{1}{z^7}+\ldots+\frac{2}{z^2}-\frac{2}{z^4}+\frac{2}{z^6}-\frac{2}{z^8}+\ldots+\frac{1}{2}-\frac{z}{4}+\frac{z^2}{8}-\frac{z^3}{16}\ldots\right]$$

$$= \frac{1}{5}\left[\ldots-2z^{-8}+z^{-7}+2z^{-6}-z^{-5}-2z^{-4}+z^{-3}+2z^{-2}-z^{-1}+\frac{1}{2}-\frac{z}{4}+\frac{z^2}{8}-\frac{z^3}{16}\ldots\right]$$

Which is the required series. **Ans.**

(*ii*) Here $|z| > 2$

$$f(z) = -\frac{1}{5}\frac{z-2}{1+z^2} + \frac{1}{5}\frac{1}{z+2} = -\frac{1}{5}\frac{1}{z^2}\frac{z-2}{1+\frac{1}{z^2}} + \frac{1}{5}\frac{1}{z}\frac{1}{1+\frac{2}{z}}$$



$$= -\frac{1}{5z^2}(z-2)\left[1+\frac{1}{z^2}\right]^{-1} + \frac{1}{5z}\left[1+\frac{2}{z}\right]^{-1}$$

$$= \frac{1}{5}\left[-\frac{1}{z}+\frac{2}{z^2}\right]\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\ldots\right] + \frac{1}{5z}\left[1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+\frac{16}{z^4}+\ldots\right]$$

$$= \frac{1}{5}\left[-\frac{1}{z}+\frac{1}{z^3}-\frac{1}{z^5}+\frac{1}{z^7}+\ldots+\frac{2}{z^2}-\frac{2}{z^4}+\frac{2}{z^6}-\frac{2}{z^8}+\ldots+\frac{1}{z}-\frac{2}{z^2}+\frac{4}{z^3}-\frac{8}{z^4}+\ldots\right]$$

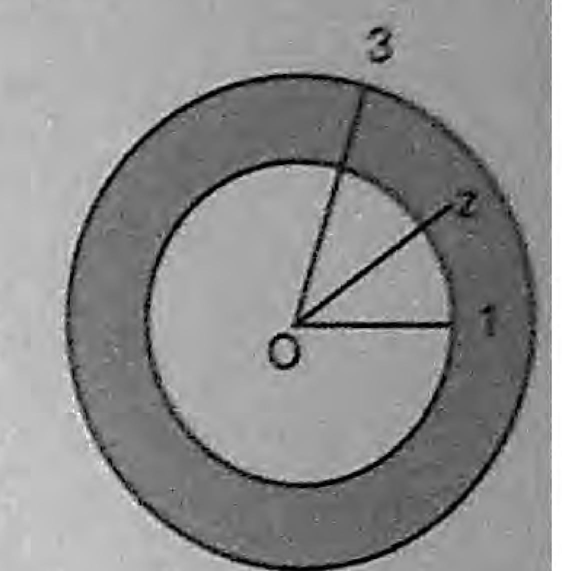
Which is the required series. **Ans.**

Example 24. *Expand* $f(z) = \dfrac{1}{(z+1)(z+3)}$ *in Laurent series valid for*

(*i*) $1 < |z| < 3$ (*ii*) $|z| > 3$

(*iii*) $0 < |z+1| < 2$ (*iv*) $|z| < 1$

Solution. $f(z) = \dfrac{1}{(z+1)(z+3)}$



$$= \frac{1}{2}\left(\frac{1}{z+1} - \frac{1}{z+3}\right)$$

(*i*) $1 < |z| < 3. \Rightarrow \dfrac{1}{|z|} < 1$ and $\dfrac{|z|}{3} < 1.$

$\Rightarrow 1 < |z|$ and $|z| < 3.$

$$f(z) = \frac{1}{2}\left[\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{3\left(1+\frac{z}{3}\right)}\right] = \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}\ldots\right) - \frac{1}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}\ldots\right)\right]$$

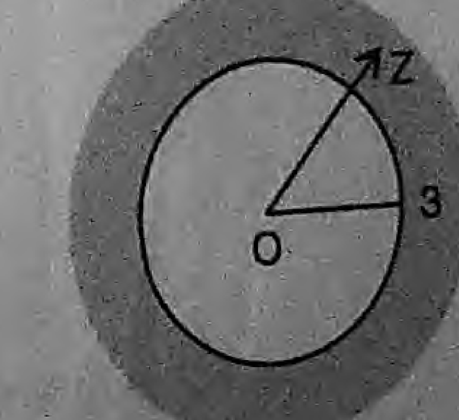
$$= \left(\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}\ldots\right) - \frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\frac{z^3}{162}\ldots$$

$$= -\frac{1}{2z^4}+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\frac{z^3}{162}\ldots$$

Which is the required series. **Ans.**

(*ii*) $|z| > 3 \Rightarrow \dfrac{3}{|z|} < 1$

$$f(z) = \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1}\right] = \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}\ldots\right) - \frac{1}{2z}\left(1-\frac{3}{z}+\frac{3^2}{z^2}\ldots\right)$$



$$= \left(\frac{1}{2z}-\frac{1}{2z}\right) + \left(\frac{-1}{2z^2}+\frac{3}{2z^2}\right) + \left(\frac{1}{2z^3}-\frac{9}{2z^3}\right) + \left(-\frac{1}{2z^4}+\frac{27}{2z^4}\right)$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{50}{z^5} + \ldots$$

Which is required expansion. **Ans.**

$$= \frac{1}{(z-a)^m}[b_m + b_{m-1}(z-a)+...b_1(z-a)^{m-1}]$$

$$= \frac{1}{(z-a)^m}\left[b_m + \sum_{n=1}^{m-1} b_n(z-a)^{m-n}\right]$$

$$\left|\sum_{n=1}^{m} b_n(z-a)^{-n}\right| = \left|\frac{1}{(z-a)^m}\left[b_m + \sum_{n=1}^{m-1} b_n(z-a)^{m-n}\right]\right|$$

$$\geq \left|\frac{1}{(z-a)^m}\right|\left\{|b_m| - \sum_{n=1}^{m-1} |b_n||z-a|^{m-n}\right\}$$

This tends to b_m $\quad |a_1 + a_2| \geq |a_1| - |a_2|$

As $z \to a$ $\quad$ R.H.S $= \infty$.

Example 28. *Write all possible Laurent series for the function*

$$f(z) = \frac{1}{z(z+2)^3}$$

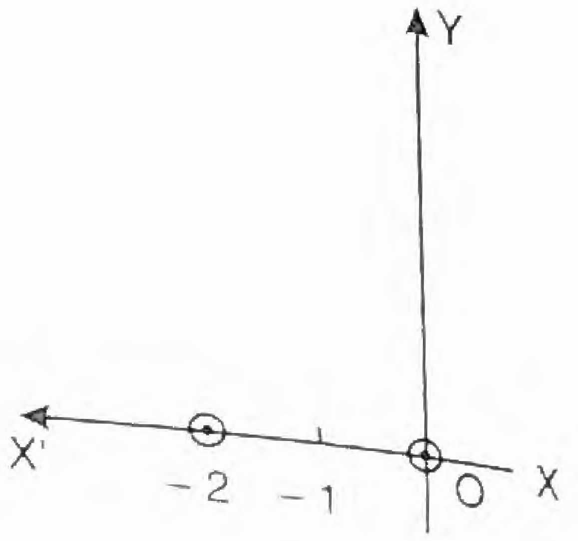
about the pole $z = -2$. Using appropriate Laurent series

Solution. To expand $\frac{1}{z(z+2)^3}$ about $z = -2$, *i.e.*, in powers of $(z+2)$, we put $z + 2 = t$.

Then $\quad f(z) = \frac{1}{z(z+2)^3} = \frac{1}{(t-2)t^3} = \frac{1}{t^3}\cdot\frac{1}{t-2}$

$$= \frac{1}{t^3}\cdot\frac{1}{-2}\cdot\left(\frac{1}{1-\frac{t}{2}}\right) = -\frac{1}{2t^3}\left(1-\frac{t}{2}\right)^{-1}$$

$$[0 < |z-2| < 1 \text{ or } 0 < |t| < 1]$$



$$= -\frac{1}{2t^3}\left[1+\frac{t}{2}+\frac{t^2}{4}+\frac{t^3}{8}+\frac{t^4}{16}+\frac{t^5}{32}+...\right]$$

$$= -\frac{1}{2t^3}-\frac{1}{4t^2}-\frac{1}{8t}-\frac{1}{16}-\frac{t}{32}-\frac{t^2}{64}+....$$

$$= -\frac{1}{2(z+2)^3}-\frac{1}{4(z+2)^2}-\frac{1}{8(z+2)}-\frac{1}{16}-\frac{z+2}{32}-\frac{(z+2)^2}{64}+...$$

Ans.

Example 29. *Expand* $\frac{e^z}{(z-1)^2}$ *about* $z = 1$

Solution. Let. $f(z) = \frac{e^z}{(z-1)^2} = \frac{e^{1+t}}{t^2}$ $\qquad$ [Put $z - 1 = t \Rightarrow z = 1 + t$]

$$= \frac{e \cdot e^t}{t^2} = \frac{e}{t^2}\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...\right] = e\left[\frac{1}{t^2}+\frac{1}{t}+\frac{1}{2!}+\frac{t}{3!}...\right]$$

$$= e\left[\frac{1}{(z-1)^2}+\frac{1}{z-1}+\frac{1}{2!}+\frac{z-1}{3!}+...\right]$$

Which is required expansion.

Ans.

Example 30. *Expand* $f(z) = \cosh\left(z+\frac{1}{z}\right)$

Solution. $f(z)$ is analytic except $z = 0$

$f(z)$ can be expanded by Laurent theorem.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty}\frac{b_n}{z^n}$$

where $\quad a_n = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-0)^{n+1}},\quad b_n = \frac{1}{2\pi i}\int_c f(z)z^{-n-1}dz$

Now, $\quad a_n = \frac{1}{2\pi i}\int_c \frac{\cosh\left(z+\frac{1}{z}\right)}{z^{n+1}}dz$ $\qquad \left[\begin{array}{l} z = \cos\theta + i\sin\theta \\ \frac{1}{z} = \cos\theta - i\sin\theta \end{array}\right]$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\frac{\cosh(2\cos\theta)\, ie^{i\theta}}{e^{i(n+1)\theta}}d\theta \qquad \left[\begin{array}{l} z = e^{i\theta} \\ dz = ie^{i\theta}d\theta \end{array}\right]$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}\frac{\cosh(2\cos\theta).ie^{i\theta}}{e^{i(n+1)\theta}}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cosh(2\cos\theta)e^{-ni\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi}\cosh(2\cos\theta)\,(\cos n\theta - i\sin n\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cosh(2\cos\theta)\cos n\theta\, d\theta - \frac{i}{2\pi}\int_c \cosh(2\cos\theta)\sin n\theta\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cosh(2\cos\theta)\cos n\theta\, d\theta + 0$$

Similarly, $\quad b_n = \frac{1}{2\pi}\int_0^{2\pi}\cosh(2\cos\theta)\cos(n\theta)\,d\theta$

If we write $\frac{1}{z}$ for z in the given function

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$$

Ans.

Example 31. Expand $\quad f(z) = \sin\left\{c\left(z+\frac{1}{z}\right)\right\}$

Solution. $f(z)$ is not analytic at $z = 0$.

Therefore $f(z)$ can be expanded by Laurent theorem.

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty}\frac{b_n}{z^n}$$

where $\quad a_n = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{(z-0)^{n+1}}$ and $b_n = \frac{1}{2\pi i}\int_c f(z)z^{n-1}dz$

Now, $\quad a_n = \frac{1}{2\pi i}\int_c \frac{\sin c\left(z+\frac{1}{z}\right)dz}{z^{n+1}}$ $\qquad \left[\begin{array}{l} z = e^{i\theta} \\ dz = ie^{i\theta}d\theta \end{array}\right]$